PHASE STRUCTURE OF THERMAL QED BASED ON THE HARD THERMAL LOOP IMPROVED LADDER DYSON-SCHWINGER EQUATION –A "GAUGE INVARIANT" SOLUTION–

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Based on the hard-thermal-loop resummed improved ladder Dyson-Schwinger equation for the fermion mass function, we study how we can get the gauge invariant solution in the sense it satisfies the Ward identity. Properties of the "gauge-invariant" solutions are discussed.

Keywords: Thermal QED, Hard Thermal Loop, Dyson-Schwinger equation, Gauge invariance.

1. Introduction and summary

The Dyson-Schwinger equation (DSE) is a powerful tool to investigate with the analytic procedure the nonperturbative structure of field theories, such as the phase structure of gauge theories. However, the full DSEs are coupled integral equtions for several unknown functions, thus are hard to be solved without introducing appropriate approximations. We usually adopt the step-by-step approach to this problem, firstly approximate the integration kernel by the tree, or, ladder kernel, next use the improved ladder one, etc. Advantage of the DSE analysis lies in the possibility of such an systematic improvement through the analytic investigation.

Analyses of the DSE have proven to be successful in studying the phase structure of vacuum gauge theories [1-3]. In the Landau gauge DSE with the ladder kernel for the fermion mass function in the vacuum QED, the fermion wave function renormalization constant is guaranteed to be unity [1], satisfying the Ward identity. Thus irrespective of the problem of the ladder approximation, the results obtained would be gauge invariant.

Same analyses have been carried out in the finite temperature/density case with the ladder kernel [4-6], and with the hard-thermal-loop (HTL) resummed improved ladder kernel [7]. Results of Ref. [7] show that at finite temperature/density it is important to correctly analyse the physical mass function Σ_R, the mass function of the "unstable" quasiparticle in thermal field theories, and also to correctly take the dominant thermal effect into the interaction kernel.

All the preceding analyses [4-7], however, suffer from the serious problem coming from the ladder approximation of the interaction kernel. Although in the vacuum case, despite the use of ladder kernel, in the analysis in the Landau gauge the Ward identity is guaranteed to be satisfied, at finite temperature /density there is no such guarantee. In fact, even in the Landau gauge the fermion wave function renormalization constant largely deviates from unity [7,8], being not even real. At finite temperature/density the results obtained from the ladder DSE explicitly violate the Ward identity, thus depend on the gauge, their physical meaning being obscure.

In this paper, we worked out, in the analysis of the HTL resummed improved ladder DS equation for the fermion mass function in thermal QED, the procedure to get the gauge invariant solution in the sense it satisfies the Ward identity, and investigated the properties of the "gauge invariant" solution.

Results of the present analysis are summarized as follows:

(1) We can determine the solution that satisfies the Ward identity, namely the fermion wave function renormalization constant being almost equal to unity. To get such a solution it is essential that the gauge parameter ξ depends on the momentum of the gauge boson.

(2) The chiral phase transition in the massless thermal QED is confirmed to occur through the second order transition.

(3) Two critical exponents ν and η are consistent with constant within the range of values of temperatures and couplingconstants under analysis: $\nu = 0.395, \eta = 0.518$.

(4) The effect of thermal fluctuation on the chiral symmetry breaking and/or restoration is smaller than that expected in the previous analysis in the Landau gauge [7].

2. The Dyson-Schwinger equation for the fermion self-energy function Σ_R

The fermion self-energy function Σ_R appearing in the fermion propagator

$$S_R(P) = [\not{P} + i\epsilon\gamma^0 - \Sigma_R(P)]^{-1} \tag{1}$$

can be decomposed at finite temperature and/or density as

$$\Sigma_R(P) = (1 - A(P))p_i\gamma^i - B(P)\gamma^0 + C(P) \tag{2}$$

with $A(P)$, $B(P)$ and $C(P)$ being the three scalar invariants to be determined. In the present analysis, we use the HTL resummed form ${}^*G^{\mu\nu}$ for the gauge boson propagator $G^{\mu\nu}$

$${}^*G^{\mu\nu}(K) = \frac{1}{{}^*\Pi_T - K^2 - i\epsilon k_0}A^{\mu\nu} + \frac{1}{{}^*\Pi_L - K^2 - i\epsilon k_0}B^{\mu\nu} - \frac{\xi}{K^2 + i\epsilon k_0}D^{\mu\nu} \tag{3}$$

where ${}^*\Pi_{L/T}$ is the HTL resummed longitudinal/transverse photon self-energy function [9], and $A^{\mu\nu}$, $B^{\mu\nu}$ and $D^{\mu\nu}$ are the projection tensors [10],

$$A^{\mu\nu} = g^{\mu\nu} - B^{\mu\nu} - D^{\mu\nu}, B^{\mu\nu} = -\tilde{K}^\mu\tilde{K}^\nu/K^2, D^{\mu\nu} = K^\mu K^\nu/K^2, \tag{4}$$

where $\tilde{K} = (k, k_0\hat{\mathbf{k}})$, $k = \sqrt{\mathbf{k}^2}$ and $\hat{\mathbf{k}} = \mathbf{k}/k$ denotes the unit three vector along $\mathbf{k}$. The parameter ξ appearing in the term proportional to the projection tensor $D_{\mu\nu}$ represents the gauge-fixing parameter ($\xi = 0$ in the Landau gauge). This gauge term plays an important role in the present analysis.

The vertex function is approximated by the tree (point) vertex. With the instantaneous exchange approximation for the longitudinal photon mode, we get the DSEs for the three invariant functions $A(P)$, $B(P)$ and $C(P)$

$$\begin{aligned}
&-p^2[1 - A(P)] = -e^2\int\frac{d^4K}{(2\pi)^4}\Bigg[\{1 + 2n_B(p_0 - k_0)\}\mathrm{Im}[\,{}^*G_R^{\rho\sigma}(P-K)] \times \\
&\Big[\{K_\sigma P_\rho + K_\rho P_\sigma - p_0(K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0}) - k_0(P_\sigma g_{\rho 0} + P_\rho g_{\sigma 0}) + pkz g_{\sigma\rho} \\
&+2p_0k_0 g_{\sigma 0}g_{\rho 0}\}\frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2k^2 - C(K)^2} + \{P_\sigma g_{\rho 0} \\
&+P_\rho g_{\sigma 0} - 2p_0 g_{\sigma 0}g_{\rho 0}\}\frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2k^2 - C(K)^2}\Big] \\
&+\{1 - 2n_F(k_0)\}\,{}^*G_R^{\rho\sigma}(P-K)\mathrm{Im}\Big[\{K_\sigma P_\rho + K_\rho P_\sigma - p_0(K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0}) \\
&-k_0(P_\sigma g_{\rho 0} + P_\rho g_{\sigma 0}) + pkz g_{\sigma\rho} + 2p_0k_0 g_{\sigma 0}g_{\rho 0}\} \times \\
&\frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2k^2 - C(K)^2} + \{P_\sigma g_{\rho 0} + P_\rho g_{\sigma 0} \\
&-2p_0 g_{\sigma 0}g_{\rho 0}\}\frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2k^2 - C(K)^2}\Big]\Bigg]\ ,
\end{aligned} \tag{5}$$

$$
\begin{aligned}
&-B(P) = -e^2 \int \frac{d^4K}{(2\pi)^4} \Big[\{1 + 2_B(p_0 - k_0)\} \mathrm{Im}[\ ^*G_R^{\rho\sigma}(P-K)] \times \\
&\Big[\{K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0} - 2k_0 g_{\sigma 0} g_{\rho 0}\} \frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \\
&+ \{2g_{\rho 0} 2g_{\sigma 0} - g_{\sigma\rho}\} \frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \Big] \\
&+ \{1 - 2n_F(k_0)\}\ ^*G_R^{\rho\sigma}(P-K) \mathrm{Im}\Big[\frac{A(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \\
&\times \{K_\sigma g_{\rho 0} + K_\rho g_{\sigma 0} - 2k_0 g_{\sigma 0} g_{\rho 0}\} \\
&+ \frac{k_0 + B(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \{2g_{\rho 0} 2g_{\sigma 0} - g_{\sigma\rho}\} \Big] \Big] \ , \qquad (6) \\
&C(P) = -e^2 \int \frac{d^4K}{(2\pi)^4} g_{\sigma\rho} \{1 + 2_B(p_0 - k_0)\} \mathrm{Im}[\ ^*G_R^{\rho\sigma}(P-K)] \times \\
&\Big[\frac{C(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} + \{1 - 2n_F(k_0)\} \times \\
&\ ^*G_R^{\rho\sigma}(P-K) \mathrm{Im}\Big[\frac{C(K)}{[k_0 + B(K) + i\epsilon]^2 - A(K)^2 k^2 - C(K)^2} \Big] \Big] \ . \qquad (7)
\end{aligned}
$$

The function $A(P)$ is nothing but the inverse of the fermion wave function renormalization constant Z_2, thus must be unity in order to satisfy the Ward identity in the ladder DSE analysis, where the vertex function receives no renormalization effect, $Z_1 = 1$.

We must solve the above DSEs and get the solution satisfying the Ward identity $Z_2 = Z_1 (= 1)$, where $Z_2 = A(P)^{-1}$. The procedure to get the "gauge invariant" solution is as follows;

(1) Assume the nonlinear gauge such that the gauge parameter ξ to be a function of the photon momentum $K = (k_0, \mathbf{k})$, and parametrize ξ as

$$\xi(k_0, k) = \sum \xi_{mn} H_m(k_0) L_n(k), \qquad (8)$$

where ξ_{mn} are unknown parameters to be determined, H_m the Hermite functions and L_n the Laguerre functions.

(2) In solving DSEs iteratively, impose the condition $A(P) = 1$ by constraint for the input-functions at each step of the iteration.

(3) Determine ξ_{mn} so as to minimize $|A(P) - 1|^2$ for the out-put functions and find the solutions for $B(P)$ and $C(P)$.

3. Results of the analysis "gauge invariant" solution

Here we present the results obtained by allowing the gauge parameter ξ to be a complex value. Number of parameters ξ_{mn} to minimize $|A(P) - 1|^2$

is $2 \times 5 \times 2 = 20$ (i.e., $m = 1 \sim 4$ and $n = 1, 2$). All the quantities with the mass dimension are evaluated in the unit of Λ, the cut-off parameter introduced as usual to regularize the DSEs.

Now we present the solution consistent with the Ward identity, i.e., the "auge invariant" solution. Firstly in Fig.1 we show $\mathrm{Re}A(P)$. For comparison, results in the constant ξ analyses are also shown in the same figure.

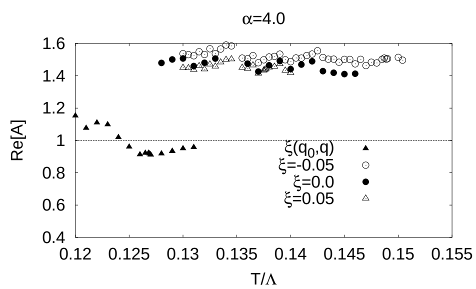

Fig. 1. Comparison of the renormalization constant $\mathrm{Re}[A]$ at the coupling constant $\alpha = 4.0$ evaluated at $p_0 = 0$, $p = 0.1$, see, text.

Next let us study the property of the phase transition. Fig.2(a) shows the real part of the fermion mass $\mathrm{Re}M(P)$, $M(P) \equiv C(P)/A(P)$, obtained from the "gauge invariant" solution, as a function of the temperature T . The mass is evaluated at $p_0 = 0$, $p = 0.1$, to be consistent with the standard prescription to define the mass in the static limit, $p_0 = 0$, $p \to 0$.

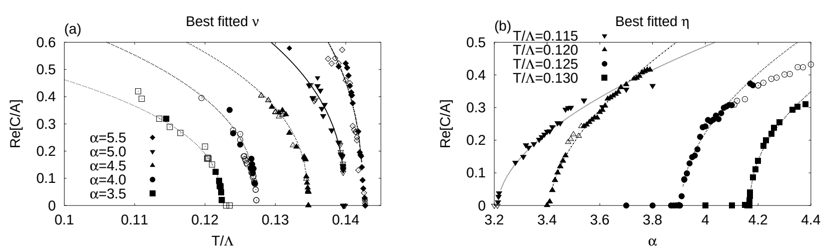

Fig. 2. (a)Temperature-dependence of the fermion mass $\mathrm{Re}[M(P)]$ for various values of the coupling α. (b) Coupling constant-dependence of $\mathrm{Re}[M(P)]$ for various values of the temperature T. Both are evaluated at $p_0 = 0$ and $p = 0.1$. As for the various curves, see text.

The curves in the figure show the best-fit curves in determining the critical temperature T_c and the critical exponent ν, by fitting, at each coupling constant α and near the critical temperature T_c , the temperature-dependent data of Re$M(P)$ to the functional form

$$\mathrm{Re}M(P) = C_T(T_c - T)^\nu. \tag{9}$$

Also shown in Fig.2(b) is the Re$M(P)$ obtained from the "gauge invariant" solution, as a function of the coupling constant α. The mass is evaluated at $p_0 = 0$, $p = 0.1$ as above. The curves in the figure show the best-fit curves in determining the critical coupling α_c and the critical exponent η, by fitting, at each temperature T and near the critical coupling α_c, the coupling-dependent data of Re$M(P)$ to the functional form

$$\mathrm{Re}M(P) = C_\alpha(\alpha - \alpha_c)^\eta. \tag{10}$$

The determined critical exponents are given in Table 1.

Table 1. Critical exponent ν for various values of the coupling constant α and critical exponent η for various values of the temperature T

α	ν	T	η
3.5	0.42800	0.115	0.54718
4.0	0.38126	0.120	0.57872
4.5	0.36420	0.125	0.51430
5.0	0.40579	0.130	0.46153

The averaged value of ν over the various coupling is $< \mu >= 0.395$, which fits to all the data Re$M(P)$ in Fig.2(a) irrespective of the value of the coupling constant. The averaged value of η over various temperatures is $< \eta >= 0.518$, which fits to all the data Re$M(P)$ in Fig.2(b) irrespective of the value of the temperature.

The phase boundary curve in the (T, α)-plane thus determined shows that the region of the symmetry broken phase shrinks to the low-temperature-strong- coupling side compared with that of the Landau gauge. This fact means that the effect of thermal fluctuation on the chiral symmetry breaking/restoration is smaller than that expected in the previous analysis in the Landau gauge [7].

4. Discussion and comments

Results presented in the present paper are preliminary, because of the rough analysis of the data processing. We are now refining the data analysis and soon get the results of the thorough reanalysis. Though the main conclusion will not be altered, several remarks should be added.

(1) Present analysis was performed by allowing the gauge parameter ξ to be a complex value. Such a choice of gauge may correspond to studying the non-hermite dynamics, thus may cause some troubles. What happens if we restrict the gauge parameter to the real value? We are studying this case, finding a remarkable result: In both cases results completely agree, thus getting a solution totally independent of the choice of gauges.

(2) In the present analysis, the consistency of the solution with the Ward identity is respected only by imposing the condition $A(P) \approx 1$. Needless to say, in solving the (improved) ladder Dyson-Schwinger equation, there are no solutions totally consistent with the Ward identity. Despite that fact, following point should be closely examined: At least around or in the static limit, $p_0 = 0$, $p \to 0$, where we calculated (defined) the mass, each invariant function $A(P)$, $B(P)$ or $C(P)$ should not have big momentum dependence. This condition may be important in connection with the consistency of the obtained solution with the gauge invariance. Result of the present analysis shows that at least $B(P)$ and $C(P)$ satisfy this condition.

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